(1) **The Uniform bounded theorem**

Statement :- Let B be a Banach space and N a normed linear space. If $\{T_i\}$ is a non-empty set of bounded (i.e. continuous) linear transformations of B into N having the property that $\{T_i(x)\}$ is a bounded subset of N for each vector x in B, then $\{\|T_i\|\}$ is a bounded set of numbers, that is $\{T_i\}$ is a subset B of (B, N).

Proof :- For each positive integer n, define

$$F_n = \{x : x \in B \text{ and } \|T_i(x)\| \leq n \text{ for all } i\} \quad \text{---(1)}$$

Then F_n is a closed subset of B as shown below

$$x \in F_n \Leftrightarrow \|T_i(x)\| \leq n \text{ for all } i$$

$$\Leftrightarrow T_i(x) \in S_n^o \text{ for all } i$$

Where S_n^o denotes the closed sphere in N with centre 0 and radius n.

$$\Rightarrow x \in T_i^{-1}[S_n^o] \text{ for all } i$$

$$\Rightarrow x \in \bigcap_i T_i^{-1}[S_n^o]$$

So that $$F_n = \bigcap_i T_i^{-1}[S_n^o]$$

Which is closed, being an intersection of closed sets.

[Since each T_i is continuous and S_n^o is closed in N, each $T_i^{-1}[S_n^o]$ is closed in B].

Further, $B = \bigcup_{n=1}^{\infty} F_n$. For if $B \neq \bigcup_{n=1}^{\infty} F_n$, then there exists some $x \in B$ such that

$x \notin F_n$ for any n

$\Rightarrow \|T_i(x)\| > n$ for all n by (1)

$\Rightarrow$ the set $\{T_i(x)\}$ is not bounded.

Which contradicts the hypothesis. Hence we must have $B = \bigcup_{n=1}^{\infty} F_n$.

So that the complete space B is the union of a sequence of its subsets. Therefore by Baire Category theorem, there exists an integer n_0 such that $\bar{F}_{n_0}$ has non-empty interior. Since F_{n_0} is closed.

$$\bar{F}_{n_0} = F_{n_0}$$

And so F_{n_0} must have non-empty interior, that is there exists some $x_0 \in F^{o}_{n_0}$ so that F_{n_0} is a nhd of x_0. Since F_{n_0} is closed, there exists a closed sphere

$$S = \{x \in B : \|x - x_0\| \leq r_0\} \subset F_{n_0} \quad \text{---(2)}$$

Now if $\|y\| < 1$, then for arbitrary but fixed i

$$\|T_i(y)\| = \|T_i(z/r_0)\| \quad \text{Where } z = r_0 y$$

$$= \frac{1}{r_0}\|T_i(z)\| = \frac{1}{r_0}\|T_i(z + x_0 - x_0)\|$$

$$= \frac{1}{r_0}\|T_i(z + x_0) - T_i(x_0)\|$$

$$\leq \frac{1}{r_0}\left[\|T_i(z + x_0)\| + \|-T_i(x_0)\|\right]$$

$$= \frac{1}{r_0}\left[\|T_i(z + x_0)\| + \|T_i(x_0)\|\right]$$

$$\leq \frac{1}{r_0}(n_0 + n_0) = \frac{2n_0}{r_0}, \text{ since } z + x_0 \text{ and } x_0 \in F_{n_0}$$

[Note that $\|z + x_0 - x_0\| = \|z\| = \|r_0 y\| = r_0\|y\| \leq r_0$ $(\because \|y\| \leq 1)$ so that $z + x_0 \in S \subset F_{n_0}$.

of Course $x_0 \in S \subset F_{n_0}$].

Thus $\|T_i(y)\| \leq \frac{2n_0}{r_0}$ if $\|y\| \leq 1$.

$\therefore \|T_i\| = \sup\{\|T_i(y)\| : \|y\| \leq 1\} \leq 2n_0/r_0$

it follows that $\{\|T_i\|\}$ is a bounded set of numbers and the proof is complete.

Theorem (B) :— ~~[illegible]~~. A non-empty subset S of a normed linear space N is bounded if and only if $f[S]$ is a bounded set of numbers for each $f \in N^*$.

Proof:— Suppose S is a bounded subset of N so that there exists a positive Constant k_1 such that

$$\|x\| \leq k_1 \text{ for all } x \in S. \quad \text{——— (1)}$$

To show that $f(S)$ is bounded for each $f \in N^*$

Now $f \in N^* \Rightarrow f$ is bounded

$\Rightarrow$ there exists $k_2 > 0$ such that $|f(x)| \leq k_2\|x\|$ for all $x \in N$ ——— (2)

From (1) and (2), we conclude that

$$|f(x)| \leq k_1 k_2 \text{ for all } x \in S.$$

Hence $f[S]$ is a bounded set of numbers for each $f \in N^*$

Conversely, let $f[S]$ be a bounded set of numbers for

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each $f \in N^*$.

To show that S is bounded. For convenience, we exhibit the vectors in S by writing $S = \{x_i\}$. We now use the natural imbedding $x_i \to F_{x_i}$ to pass from S to the corresponding subset $\{F_{x_i}\}$ of N^{**}; that F_x is defined by

$$F_x(f) = f(x)$$

Hence our assumption that

$$f[S] = \{f(x_i)\}$$

is bounded for each f is equivalent to the assumption that $\{F_{x_i}(f)\}$ is bounded set for each $f \in N^*$ and since N^* is complete, it follows from the uniform bounded theorem that $\{F_{x_i}\}$ is a bounded subset of N^{**}, that is $\{\|F_{x_i}\|\}$ is a bounded set of numbers. Since the natural imbedding preserves norms, we have

$$\|F_{x_i}\| = \|x_i\|$$

for each $x_i \in S$. It follows that $\{\|x_i\|\}$ is a bounded set of numbers, that is $S = \{x_i\}$ is a bounded subset of N, thus completing the proof.

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